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# **Considerations on the Relative Accuracy of TDOA Estimation for FSK and OOK Signalling**

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Defence Science and Technology Organisation

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## **ABSTRACT**

This report outlines accuracy benefits in the time difference of arrival (TDOA) estimation of emitters employing 2-FSK signals over a corresponding OOK signal. We consider TDOA estimation via the conventional approach, using maximization of the magnitude of the complex correlation function. By drawing an analogy with the improvement that is theoretically possible by incorporating phase information into TDOA estimation for an arbitrary signal, it is suggested that for sufficiently high SNR, substantially more accurate estimation of the TDOA is possible with an FSK signal.

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# Considerations on the Relative Accuracy of TDOA Estimation for FSK and OOK Signalling

## Executive Summary

The estimation of the relative time delay between a signal component common to that received at two spatially separated antennas is a problem of common interest. The time difference of arrival (TDOA) is usually estimated as the argument of the peak in the cross-correlation computed between the two received signals. Conventionally, to minimize the sampling rate and consequently the computational complexity, the correlation is performed on the complex envelopes of the received signals.

This report compares the TDOA estimation variance (accuracy) achievable with two popular modulation schemes: on-off keying (OOK), and 2-mode frequency shift keying (2-FSK). The former is characterised by the carrier frequency,  $f_c$ , and the bit rate  $R_b$ ; while the latter is also characterised by the frequency deviation  $f_0$  (of the two FSK components from  $f_c$ ).

The Cramer-Rao bound for the variance of the TDOA is inversely dependent on an integral (over frequency) involving the ‘SNR spectra’, the cross-correlation integration time, the signal bandwidth as well as a quadratic frequency multiplier. For the case in which the signals and noise have (piecewise) flat power spectra, the latter term leads to the TDOA estimation variance being inversely proportional to the ‘rms radian frequency’. This suggests that the 2-FSK signal type should have a relatively lower TDOA estimation variance than the OOK signal type, with the relative improvement growing with  $f_0/R_b$ . This is reflected in the complex correlation envelope by the 2-FSK signal showing a relatively sharper primary correlation peak, by a factor related to  $f_0/R_b$ . On the other hand, the complex correlation envelope of the 2-FSK signal includes an oscillatory component, leading to ambiguous secondary peaks, lying at multiples of  $1/2f_0$  from the true TDOA. Heuristically, the likelihood of such secondary peaks causing erroneous TDOA estimation increases within increasing  $f_0/R_b$  and decreasing SNR.

With the aim of obtaining a stronger understanding relating to SNR,  $f_0/R_b$  and the relative improvement in TDOA estimation provided by 2-FSK over OOK, we draw on an analogy with the TDOA estimation improvements achievable by incorporating complex correlation phase information with the complex correlation envelope. A series of theoretical studies have been published on this analogous consideration. These imply that, for the 2-FSK vs OOK comparison case, two SNR thresholds  $SNR_2 > SNR_1$  exist. In particular: (i) for  $SNR < SNR_1$ , the likelihood of ambiguity induced TDOA estimation errors for the 2-FSK signal type is substantial, leading to essentially the same TDOA estimation accuracies as that provided by OOK; (ii) for  $SNR > SNR_2$ , the likelihood of ambiguity induced TDOA estimation errors with the 2-FSK signal type is negligible, leading to this signal type showing substantially improved TDOA estimation over the OOK signal type; (iii) the SNR thresholds  $SNR_1$  and  $SNR_2$  increase as  $f_0/R_b$  increases. The report concludes with a simulation study confirming the above features. Importantly, in many applications, the required SNR ( $> SNR_2$ ) to avoid ambiguity induced errors is practically achievable, leading to the 2-FSK signal type showing an improvement (reduction) in TDOA estimation variance on the order of  $(f_0/R_b)^2$ .

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## Glossary

**CW** Continuous Wave

**DTO** Differential Time Offset

**FFT** Fast Fourier Transform

**FSK** Frequency Shift Keyed (modulation)

**IF** Intermediate Frequency

**OOK** On-Off Keyed (modulation)

**RF** Radio Frequency

**SNR** Signal to Noise Ratio

**TDOA** Time Difference Of Arrival

**FIR** Finite Impulse Response

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# 1 Introduction

The estimation of the relative delay between a signal component common to that received at two spatially separated antennas is a problem of general interest. Consider the reception of a transmitted signal  $u(t)$  by two spatially separated antennas, denoted  $r_1(t)$  and  $r_2(t)$

$$\begin{aligned} r_1(t) &= \alpha_1 u(t - T_1) + n_1(t) \\ r_2(t) &= \alpha_2 u(t - T_2) + n_2(t), \end{aligned} \quad (1)$$

where  $n_1(t)$  and  $n_2(t)$  are additive noise with arbitrary distribution. For clarity and to simplify the analysis, the scalar coefficients  $\alpha_1$  and  $\alpha_2$  will arbitrarily be set to one and  $T_1$  will be set equal to zero, so that the relative time delay, sometimes known as the Time Difference of Arrival (TDOA) or Differential Time Offset (DTO) [3],  $T = T_2$ . This is often estimated as the argument of the correlation peak  $\hat{T} = \arg \max_{\tau} R_{r_1 r_2}(\tau)$ . If one had the ability to exactly compute this cross correlation function then the approach would be optimal, but when the correlation must be estimated, the optimal (maximum likelihood) approach is to emphasise the energy associated with the signal component by appropriately pre-filtering the  $r_1(t)$  and  $r_2(t)$  in a *generalised correlation* [1, 5]. In the absence of *a priori* knowledge of the signal and noise characteristics, a pragmatic approach can be as simple as prefiltering the signals to approximately the bandwidth that contains most of the signal energy.

To minimise the sampling rate and consequently the computational complexity, correlation is often performed [5] on the complex envelopes  $\tilde{r}_1(t)$  and  $\tilde{r}_2(t)$  so that  $\hat{T} = \arg \max_{\tau} |\hat{R}_{\tilde{r}_1 \tilde{r}_2}(\tau)|$ . Consider the case where the signal and noise are restricted to within a passband bandwidth  $f_c \pm B/2$ , where  $f_c$  and  $B$  are the carrier frequency and bandwidth respectively and where the correlation is estimated over an  $L$  second interval. For sufficiently high signal to noise ratio (SNR)<sup>1</sup> the Cramer Rao bound for the variance of the estimation of time delay using complex envelopes is given by [5]

$$\sigma_{min}^2 = \left[ 4\pi^2 L \int_{f_c - B/2}^{f_c + B/2} 2SNR(f)(f - f_c)^2 df \right]^{-1}, \quad (2)$$

where

$$SNR(f) = \frac{G_{uu}(f)/G_{n1n1}(f) \cdot G_{uu}(f)/G_{n2n2}(f)}{1 + G_{uu}(f)/G_{n1n1}(f) + G_{uu}(f)/G_{n2n2}(f)} \quad (3)$$

and  $G_{uu}(f)$ ,  $G_{n1n1}(f)$  and  $G_{n2n2}(f)$  are the signal and noise auto-spectra. It can be shown [2] that for the case where both the signal and noise have white power spectra, the variance is bounded by the minimum [3]

$$\sigma_{min}^2 = \frac{1}{\beta^2 \gamma_{eq} L B}, \quad (4)$$

---

<sup>1</sup>There exists an SNR threshold region [4], below which time delay estimation contains ambiguities, but above which, the variance is bounded by the Cramer Rao low bound.

where the equivalent SNR

$$\frac{1}{\gamma_{eq}} = \frac{1}{2} \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_1 \gamma_2} \right) \quad (5)$$

embodies the SNR associated with the signal received at the two apertures,  $\gamma_1$  and  $\gamma_2$ .  $\beta$  is defined as the ‘rms radian frequency’ [3],

$$\beta = 2\pi \left[ \frac{\int_{-\infty}^{\infty} f^2 \tilde{G}_{uu}(f) df}{\int_{-\infty}^{\infty} \tilde{G}_{uu}(f) df} \right]^{1/2}, \quad (6)$$

where the  $\tilde{G}_{uu}(f) = G_{uu}(f - f_c)$ .

Although equation (4) applies to the white signal and noise case, it nevertheless suggests more general dependencies for the achievable TDOA estimation variance. In particular, it can be seen that the variance will reduce as the bandwidth of the signal increases, as might be expected. However, it doesn’t specifically reveal whether there are benefits associated with any particular modulation format. In the next section, an analysis will be presented that makes clear the variance reduction that could be achieved by conducting the correlation estimate at an RF frequency. Section 3 leverages from that analysis to suggest that FSK modulation exhibits a property that provides a TDOA estimate that can be more accurate than a comparable OOK modulation. The accuracy improvement is illustrated through simulations.

## 2 Passband TDOA estimation

Although it is common for TDOA to be estimated from the computation of the cross correlation between two complex baseband signals, it is theoretically possible to obtain an improved estimate by instead computing the cross correlation between the signals at their received RF frequencies. At sufficiently high SNR, the estimation variance is governed by the Cramer Rao bound, which for a real, passband signal with  $f_c$  Hz carrier frequency, is given by [5]

$$\sigma_{min}^2 = \left[ 4\pi^2 L \int_{f_c - B/2}^{f_c + B/2} 2SNR(f) f^2 df \right]^{-1}. \quad (7)$$

In comparing (7) to (2), it can be seen that the bound differs only by the inclusion of  $f^2$  in lieu of  $(f - f_c)^2$ . For complex processing, the variance depends only on the frequency spread of the signal about its carrier frequency, whereas for real processing, the variance is reduced as the SOI frequency content increases. In sections 2.1 and 2.2 an analysis is presented that provides insight into the reason for this.

### 2.1 Complex baseband TDOA estimation

Consider the bandpass representation of the signal

$$\begin{aligned} u(t) &= \Re[s(t) \exp(j2\pi f_c t)] \\ &= a(t) \cos(2\pi f_c t) - b(t) \sin(2\pi f_c t), \end{aligned} \quad (8)$$

where  $s(t) = a(t) + jb(t)$  and which is assumed to be transmitted from a fixed location. For the ensuing analysis, we neglect the noise components on the received signals, mindful that these will give rise to terms that only distract from the observations. Complex baseband TDOA estimation considers the complex quasi-basebanded, downconverted signals:

$$r_1(t) \longrightarrow x_1(t) = s(t) \exp(j2\pi f_I t) \quad (9)$$

$$r_2(t) \longrightarrow x_2(t) = s(t - T) \exp(j2\pi f_I t) \exp(-j2\pi f_c T) \quad (10)$$

where  $f_I$  is the quasi-basebanded intermediate frequency. Estimation of the TDOA is via maximisation of the magnitude of the correlation function:

$$\begin{aligned} \hat{T} &= \arg \max_{\tau} |R(\tau)| \\ &= \arg \max_{\tau} \left| \int_t x_1(t - \tau) x_2^*(t) dt \right| \\ &= \arg \max_{\tau} \left| \exp(j2\pi(f_c T - f_I \tau)) \int_t s(t - \tau) s^*(t - T) dt \right| \\ &= \arg \max_{\tau} \left| \int_t s(t - \tau) s^*(t - T) dt \right| \end{aligned} \quad (11)$$

As an example, consider an emitter employing OOK modulation such that  $s(t) = a(t)$  is a binary unipolar NRZ signal, with amplitude  $A$  and bit rate  $R_b$ . For simplicity, we set  $A = 1$ . Then the TDOA estimate is from (11),

$$\hat{T} = \arg \max_{\tau} \left| \int_t a(t - \tau) a(t - T) dt \right|. \quad (12)$$

## 2.2 Real Passband TDOA Estimation

The complex baseband approach from section 2.1 ignores the phase information of  $R(\tau)$ . One approach to exploiting the phase information is to process the real passband collected signals which again, neglecting the noise terms can be represented as

$$r_1(t) = a(t) \cos(2\pi f_c t) - b(t) \sin(2\pi f_c t) \quad (13)$$

$$r_2(t) = a(t - T) \cos(2\pi f_c (t - T)) - b(t - T) \sin(2\pi f_c (t - T)). \quad (14)$$

Estimation of the TDOA is then via

$$\begin{aligned} \hat{T} &= \arg \max_{\tau} R(\tau) \\ &= \arg \max_{\tau} \int_t r_1(t - \tau) r_2(t) dt \\ &= \arg \max_{\tau} [\cos(2\pi f_c (T - \tau)) \int_t a(t - \tau) a(t - T) + b(t - \tau) b(t - T) dt \\ &\quad + \sin(2\pi f_c (T - \tau)) \int_t a(t - \tau) b(t - T) - b(t - \tau) a(t - T) dt], \end{aligned}$$

where terms at double the carrier frequency have been assumed to integrate out. It is again useful to consider the OOK example from section 2.1. In this case, the TDOA estimate simplifies to

$$\hat{T} = \arg \max_{\tau} \left[ \cos(2\pi f_c (T - \tau)) \int_t a(t - \tau) a(t - T) \right] dt. \quad (15)$$

The additional “phase related” terms  $\cos(2\pi f_c(T-\tau))$  and  $\sin(2\pi f_c(T-\tau))$  potentially lead to significantly improved TDOA estimation, due to a modulation sharpening effect on the peak of the correlation function. The improvement is dependent on the ratio of  $f_c/B$ . These “phase” terms however, also introduce estimation ambiguities:

$$\hat{T} = T + n/f_c \quad (16)$$

where  $n$  is an integer. The susceptibility to these ambiguities increases as  $f_c/B$  increases and as the SNR and the correlation integration time ( $L$ ) decreases.

As an illustration, consider the case where the signal and noise *are both white* within  $f_c \pm B/2$ . For sufficiently strong SNR, satisfying [4, 5]

$$SNR > 6f_c^2/(\pi^2 B^3 L)[\phi^{-1}(B^2/(24f_c^2))]^2 \quad (17)$$

$$\text{where } \phi(y) = 1/\sqrt{2\pi} \int_y^\infty \exp(-x^2/2)dx, \quad (18)$$

the improvement (reduction) in TDOA estimation variance (using the real passband processing approach rather than the conventional complex baseband processing approach), is given by [5]

$$\sigma_{T,conv}^2/\sigma_T^2 = 1 + 12(f_c/B)^2. \quad (19)$$

Although not exactly applicable to more general spectral shapes, these expressions nevertheless suggest the order of improvement that can be obtained by estimating the TDOA using the passband approach. It should be noted however, that this improvement assumes operation in the SNR region governed by the Cramer Rao bound and that any ambiguity induced TDOA estimates are clearly detected as outliers and are ignored. As the SNR falls (e.g. for the case of bandlimited white noise and signal, below the threshold of (17)), the relative improvement decreases due to the increased influence of ambiguity induced TDOA errors.

For sufficiently poor SNR (e.g. for the case of bandlimited white noise and signal),

$$SNR < 2.76f_c^2/(\pi^2 B^3 L) \quad (20)$$

the influence of the TDOA ambiguities dominates and the two approaches lead to essentially the same TDOA estimation variance.

To re-iterate, the real passband processing approach provides, at worst (corresponding to ‘poor’ SNR conditions as described by equation (20), TDOA estimation accuracies equal to that provided by the conventional complex baseband processing approach. For sufficiently ‘good’ SNR conditions, the real passband processing approach provides improved accuracy.

Simulations were performed to conduct a spot check on the validity of equation (19) and the extent of its applicability beyond a bandlimited white signal to a modulated waveform such as OOK. For this purpose, the following parameters were considered: an SNR equal to -10 dB within a  $B = 10\text{kHz}$  signal bandwidth, a 1.5 sec integration period and  $f_s = 312.5\text{ kHz}$ . With a carrier frequency  $f_c = 5\text{ kHz}$ , the respective rms TDOA for real passband and complex TDOA estimation was  $1.9\mu\text{sec}$  and  $3.2\mu\text{sec}$ . The improvement in simulation variance is then approximately a ratio of 2.8, whereas equation (19) would

suggest a ratio equal to 4.0. At  $f_c = 10$  kHz the rms TDOA for real passband estimation was  $0.9\mu\text{sec}$  while rms TDOA for complex estimation was again  $3.2\mu\text{sec}$ . In this case, the variance ratio was approximately 12.6, closely agreeing with 13.0 as suggested by (19). These results, although by no means conclusive, give confidence to the validity of equation (19).

Now consider simulations using an OOK modulated signal, bandlimited to its baud rate frequency  $f_b = 5$  kHz, so that  $B = 10$  kHz again. Under the same conditions as the previous simulations for the white signal, with  $f_c = 5$  kHz, the rms TDOA for the real passband and complex estimation was  $2.4\mu\text{sec}$  and  $14.3\mu\text{sec}$  respectively<sup>2</sup>. The variance ratio was consequently approximately 35.5, substantially greater than for the bandlimited white signal. With  $f_c = 10\text{kHz}$ , the rms TDOA for real and complex estimation was  $0.96$  and  $14.3\mu\text{sec}$  respectively, with a variance ratio approximately equal to 222. Clearly, the degree of improvement associated with an OOK modulated signal is substantially greater than with a bandlimited white signal.

The reason that the degree of TDOA accuracy improvement obtained through estimating the correlation function using the real, passband data is substantially better for a bandlimited OOK signal than for a comparable bandlimited white noise signal relate specifically to the shape of the correlation function of the message in each case. In particular, the correlation function of a bandlimited square waveform (the message of an OOK signal) is broader than for a bandlimited white noise waveform<sup>3</sup>. Because of this greater broadness, the sharpening effected by the modulation has more impact than for the bandlimited white noise signal.

In Appendix A it is shown that the improvement obtained through computing the correlation function from the passband data can be alternatively obtained through a more explicit incorporation of phase information in a modified complex baseband approach. It needs to be emphasised that either mechanism to achieve variance reduction through the exploitation of phase information requires a sampling rate  $f_s \geq 2f_c$  which, typically, is significantly higher than the sampling rate,  $f_s \geq 2B$ , required with the conventional complex baseband approach.

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<sup>2</sup>Note that at this SNR and with these parameters, performance is at the edge of the region bounded by the Cramer Rao bound and anomalous correlation peaks start to be observed in the simulations. Consequently, TDOA estimates were hard limited to between  $\pm 10$  samples from the true peak, to ensure the results can be interpreted as applicable to the Cramer Rao bounded region.

<sup>3</sup>Specifically, first consider the OOK signal. The square message has a power spectrum defined by a sinc squared shape, with first nulls at  $\pm B/2$ . Its autocorrelation function is the inverse Fourier transform, which is a triangular function that tapers to zero at  $\tau = \pm 2/B$ . The bandlimiting of the OOK signal constitutes multiplication of its power spectrum with a square filter, extending to  $\pm B/2$  Hz, which has an inverse Fourier transform that is a sinc function with first nulls at  $\pm 1/B$ . Consequently, the autocorrelation function of the bandlimited OOK message is spread by the convolution of the sinc and triangular function. The bandlimited white noise signal, on the other hand, has an autocorrelation function that is simply that same sinc function.

### 3 Comparison between TDOA estimation accuracy for OOK and FSK modulation

In section 2.2 it was shown that for an arbitrary signal, if the TDOA is estimated using a passband correlation, the resulting correlation function can in some circumstances be sharpened by an inherent oscillatory term. In this section, it will be shown that for the specific case of an FSK signal, an analogous sharpening can also be achieved, relative to an OOK signal, even when using a conventional complex baseband correlation estimate.

Consider an emitter employing a 2-FSK signal formed as a composite of two phase-inverted OOK signals:

$$\begin{aligned}
 u(t) &= m(t) \cos(2\pi(f_c + f_o)t) + (1 - m(t)) \cos(2\pi(f_c - f_o)t - \phi) \\
 &= m(t) \cos(2\pi f_o t) \cos(2\pi f_c t) - m(t) \sin(2\pi f_o t) \sin(2\pi f_c t) \\
 &\quad + (1 - m(t)) \cos(2\pi f_o t + \phi) \cos(2\pi f_c t) + (1 - m(t)) \sin(2\pi f_o t + \phi) \sin(2\pi f_c t) \\
 &= \text{Re}[s(t) \exp(j2\pi f_c t)]
 \end{aligned}$$

where  $\phi$  is an arbitrary phase offset,  $m(t)$  is a binary, unipolar, NRZ message and

$$s(t) = m(t) \exp(j2\pi f_o t) + (1 - m(t)) \exp(-j2\pi f_o t - j\phi). \quad (21)$$

The signal is again collected by two stationary apertures, as in equation (1). Complex baseband estimation of the TDOA,  $T$  is achieved via the downconverted signals which, again ignoring noise contributions are

$$\begin{aligned}
 r_1(t) &\longrightarrow x_1(t) = s(t) \exp(j2\pi f_I t) \\
 r_2(t) &\longrightarrow x_2(t) = s(t - T) \exp(j2\pi f_I t) \exp(-j2\pi f_c T).
 \end{aligned} \quad (22)$$

The TDOA is estimated using

$$\begin{aligned}
 \hat{T} &= \arg \max_{\tau} |R(\tau)| \\
 &= \arg \max_{\tau} \left| \int_t x_1(t - \tau) x_2^*(t) dt \right| \\
 &= \arg \max_{\tau} \left| \exp(j2\pi(f_c T - f_I \tau)) \int_t s(t - \tau) s^*(t - T) dt \right|.
 \end{aligned} \quad (23)$$

The exponential term in (23) has no effect on the magnitude, so can be omitted. Further, the integrand can be expanded as

$$\begin{aligned}
 s(t - \tau) s^*(t - T) &= (m(t - \tau) \exp(j2\pi f_o(t - \tau)) + (1 - m(t - \tau)) \exp(-j2\pi f_o(t - \tau) - j\phi)) \\
 &\quad \times (m(t - T) \exp(j2\pi f_o(t - T)) + (1 - m(t - T)) \exp(-j2\pi f_o(t - T) - j\phi))^* \\
 &= (m(t - \tau) \exp(j2\pi f_o(t - \tau)) + (1 - m(t - \tau)) \exp(-j2\pi f_o(t - \tau) - j\phi)) \\
 &\quad \times (m(t - T) \exp(-j2\pi f_o(t - T)) + (1 - m(t - T)) \exp(j2\pi f_o(t - T) + j\phi)) \\
 &= m(t - \tau) m(t - T) \exp(j2\pi f_o(T - \tau)) \\
 &\quad + (1 - m(t - \tau))(1 - m(t - T)) \exp(-j2\pi f_o(T - \tau)) \\
 &\quad + m(t - \tau)(1 - m(t - T)) \exp(j2\pi 2f_o t) \exp(-j2\pi(T + \tau) + j\phi) \\
 &\quad + m(t - T)(1 - m(t - \tau)) \exp(-j2\pi 2f_o t) \exp(j2\pi(T + \tau) - j\phi)
 \end{aligned}$$



If it is assumed that the terms corresponding to double the FSK frequency deviation integrate out, then the integral simplifies to

$$\begin{aligned}
\int_t s(t-\tau)s^*(t-T)dt &= \int_t m(t-\tau)m(t-T)\exp(j2\pi f_o(T-\tau)) \\
&\quad + (1-m(t-\tau))(1-m(t-T))\exp(-j2\pi f_o(T-\tau))dt \\
&= \int_t m(t-\tau)m(t-T)(\exp(j2\pi f_o(T-\tau)) + \exp(-j2\pi f_o(T-\tau))) \\
&\quad + (1-m(t-\tau)-m(t-T))\exp(-j2\pi f_o(T-\tau))dt \\
&= 2\cos(2\pi f_o(T-\tau)) \int_t m(t-\tau)m(t-T) \\
&\quad + \exp(-j2\pi f_o(T-\tau)) \int_t (1-m(t-\tau)-m(t-T))dt.
\end{aligned}$$

If  $m(t)$  has a balanced distribution of logical '1's and '0's then the integration of the term  $1-m(t-\tau)-m(1-T)$  approaches zero for  $L \gg T, \tau$ . Consequently, the integral simplifies further to

$$\int_t s(t-\tau)s^*(t-T)dt = 2\cos(2\pi f_o(T-\tau)) \int_t m(t-\tau)m(t-T)dt$$

and the TDOA estimate is

$$\hat{T} = \arg \max_{\tau} \left| \left( 2\cos(2\pi f_o(T-\tau)) \int_t m(t-\tau)m(t-T)dt \right) \right|. \quad (24)$$

This has a structure that is very similar to equation (15) for generic real passband TDOA estimation, with the correlation magnitude  $|R(\tau)|$  gain maximized by  $\hat{T} = \tau = T$ . In an analogous manner to the generic passband processing, the presence of the  $\cos(\cdot)$  term increases the curvature (or "sharpness") at the peak value (at  $\hat{T} = \tau = T$ ), which can lead to enhanced TDOA estimation accuracy. Instead of being dependent on  $f_c$ , for the present case the curvature will depend on  $f_0$ . The magnitude of this increase in curvature will then depend on the factor  $f_o/B$ , where  $B$  is a measure of the signal bandwidth which, drawing from the analogy between (15) and (24) is suggested as  $B = 2f_b$ .<sup>4</sup>

The discussions of the preceding paragraph have exploited the similarities between (15) and (24) and have ignored the differences. One such difference is the presence of the factor '2' in the expression, which represents a 3dB greater correlation peak than the OOK peak correlation value, but it should be appreciated that this is due to the FSK signal having double the power. The other significant difference in the expressions is the presence of the magnitude operation in (24). One of the clearest consequences of the magnitude operation will be a dependence on the double frequency term,  $2f_0$ , due to its rectifying action. Consequently, the oscillatory nature of the  $\cos(\cdot)$  term will introduce potential TDOA estimation ambiguities at

$$\hat{T} = T \pm n/2f_o$$

<sup>4</sup>The analysis for the generic passband TDOA estimation assumes the signal and noise are both white within the band  $f_c \pm B/2$ . An exact analogy between that analysis and the present would require white signal and noise between  $f_0 \pm B/2$ .

where  $n$  is an integer. The susceptibility to these ambiguities increases as  $f_o/B$  increases and as the SNR and the correlation integration time ( $L$ ) decreases. However, unlike for the generic passband case, the sharpening that accompanies the FSK signal is dependent on the frequency deviation and not the RF carrier frequency. Consequences of this relationship include it being more likely that the performance improvement is realisable in practise, as well as having substantially less demanding sampling rate requirements.

By drawing on the similarities between equations (15) and (24)<sup>5</sup>, and with analogy to (17), a rule of thumb could be suggested that for sufficiently strong SNR, satisfying:

$$SNR > 6f_o^2/(\pi^2 B^3 L)[\phi^{-1}(B^2/(24f_o^2))]^2 \quad (25)$$

$$\text{where } \phi(y) = 1/\sqrt{2\pi} \int_y^\infty \exp(-x^2/2)dx \quad (26)$$

the 2-FSK signal would provide an improved (reduced) TDOA estimation variance (over the OOK signal) given by:

$$\sigma_{T,OOK}^2/\sigma_{T,2FSK}^2 = 1 + 12(f_o/B)^2. \quad (27)$$

It needs to be acknowledged that the OOK and FSK signals will generally have power spectra that differ significantly from being white, so it is expected that it will not be possible to apply equation (27) exactly in practice. It should also be noted that this improvement assumes that any ambiguity induced TDOA estimates are clearly detected as outliers and are ignored. As the SNR falls below the threshold of (25), the relative improvement decreases due to the increased likelihood that not all ambiguity induced TDOA estimates will be ignored.

For sufficiently small SNR, the two signal types lead to essentially the same TDOA estimation variance. With analogy to equation (20) for the real passband TDOA process, a rule of thumb could be suggested that the 2-FSK signal will lead to improved TDOA estimation accuracy if

$$SNR > 2.76f_o^2/(\pi^2 B^3 L). \quad (28)$$

### 3.1 Illustration through simulation results

The observations of the previous sections are illustrated here through Matlab simulation results. Unless otherwise stated, the results were generated with with arbitrarily chosen parameters:  $f_s = 312.5$  kHz,  $f_b = 5$  kHz,  $f_o = 10$  kHz,  $L = 1.5$  sec and a carrier frequency randomly chosen within the range 10 Hz to 333 Hz. The bandwidths of the signals and noise were constrained to approximately  $\pm f_b$  Hz. The TDOA was randomly selected within a  $\pm 1000$  sample range and 100 iterations were used in each trial. The SNR is measured within the signal bandwidth. That is, approximately  $f_c \pm f_b$  for OOK and a  $2f_b$  band centered at both  $f_c \pm f_o$  for FSK.

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<sup>5</sup>Substituting  $f_o$  instead of  $f_c$ . It could be conjectured that  $2f_o$  would be more appropriate, given the rectification effect of the magnitude operator in (24). However the rectification, although giving rise to double frequency components, will not change the shape of the modulating cosine term associated with the main correlation lobe.

Figures 1 and 2 show a central truncation of the correlation functions computed for a particular, arbitrary data set, for OOK and FSK signals with both 5 kHz and 20 kHz frequency deviation. The figures correspond respectively to a received SNR of +60dB and -9dB, on both signals. The sharpening of the main correlation peak for FSK with respect to OOK is clear. However, particularly for the -9dB case, it is also clear that there is an ambiguity associated with the selection of the ‘correct’ correlation peak.

The relative rms TDOA estimation accuracy between OOK and FSK is shown in Figures 3 and 4. Figure 3 shows the performance over the approximate received SNR range -14 dB to 0 dB and shows a considerable accuracy advantage with FSK. Previous experience with other OOK simulations suggest that for this parameter set, the detection of anomalous TDOA peaks begins to become noticeable below -12 dB<sup>6</sup>, so this represents the lower SNR end of useful TDOA estimation.

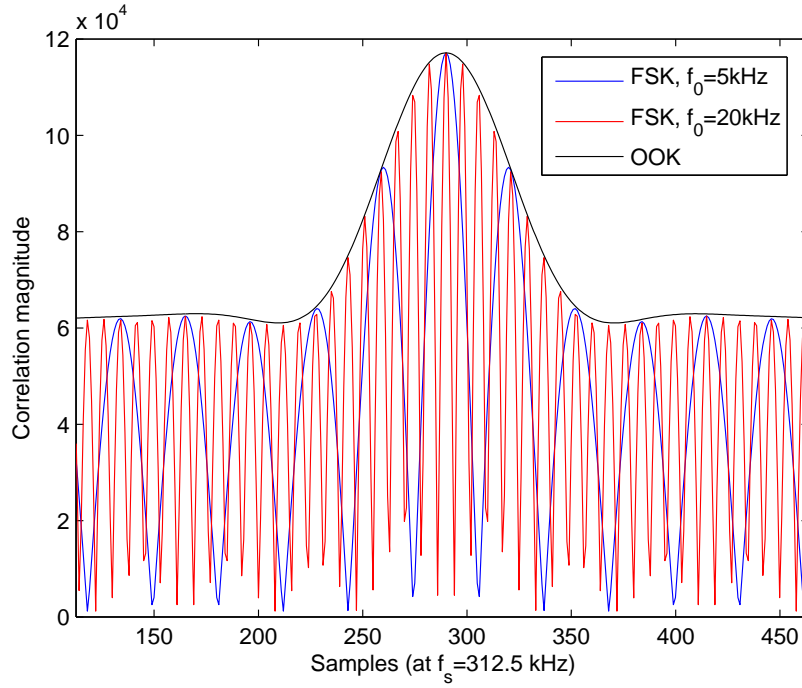
At higher SNR levels, smaller deviations are associated with the TDOA estimation, so a faster sampling rate is required to resolve such performance. Figure 4 shows the simulation results with  $f_s = 10$  MHz. In order to achieve computationally tractable Matlab simulations, the integration period was shortened to  $L = 0.15$  sec in association with the higher sampling rate. This reduces the post-integration effective SNR, so that the estimation variance at 0 dB SNR for  $L = 1.5$  sec should roughly equate to that at 8.5 dB SNR with  $L = 0.15$  sec<sup>7</sup>. Another practicality with the use of  $f_s = 10$  MHz is that it is intractable to use a bandlimiting FIR filter response that accurately restricts the message bandwidth to  $\pm f_b$  Hz. A consequence of this is that although representative relative accuracy between OOK and FSK is shown in Figure 4 there will be some absolute error. Nevertheless, comparison of the OOK TDOA accuracy at 8.5 dB in Figure 4 and 0 dB in Figure 3 reveals fairly close consistency.

Figure 4 is depicted with a log vertical scale to better accommodate the full range of TDOA estimation accuracies encountered. It shows a fairly constant ratio between the rms TDOA accuracies for OOK and FSK estimation at high SNR, with the accuracy improving with increasing frequency deviation  $f_0$ . However at low SNR, an almost reverse relationship is evident. Both of these observations are intuitively satisfying; at high SNR one expects the accuracy to improve with increasing frequency deviation due to the sharpening of the correlation function, whereas the associated ambiguities in the correlation function are closer in both time and magnitude to the correct peak, leading to poorer low SNR performance.

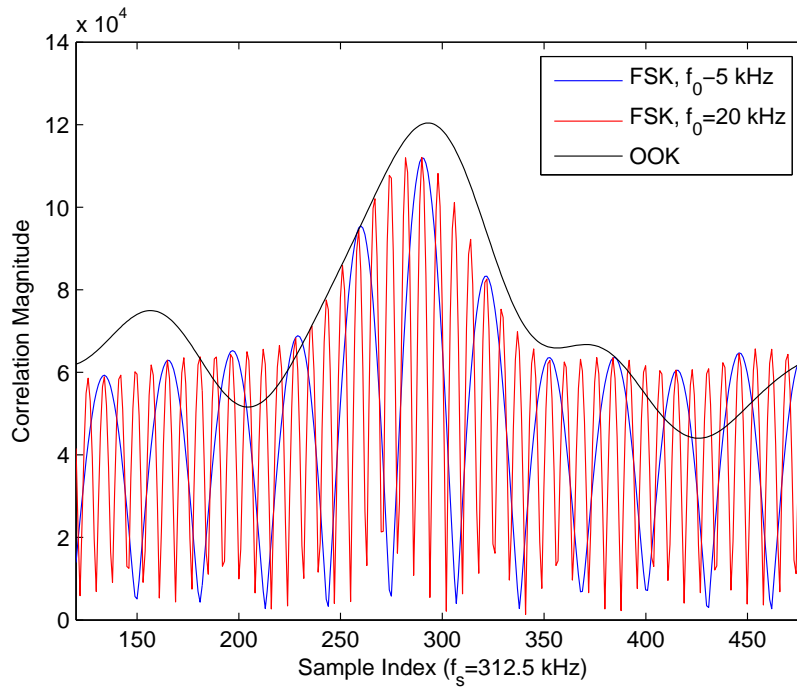
Prior to considering the quantitative comparison between OOK and FSK performance, it should be first considered that the rms TDOA shown in Figure 4 is plotted against SNR within the signal bandwidth, which is approximately 10 kHz for the OOK signal and twice this for FSK. Computed at +8.5 dB SNR, the reduction in rms TDOA achieved in simulation through using FSK in lieu of OOK is 8.4, 15.2, 27.3 and 80.0, for FSK

<sup>6</sup>That is, spurious peaks located some distance from the correct peak are sometimes encountered in one or more simulation runs out of 100.

<sup>7</sup>Equation (4) shows that for the case where the signal and noise are white, a reduction in  $L$  can be offset by a corresponding increase in equivalent SNR  $\gamma$ . Application of equation (5) allows the corresponding SNR to be evaluated. Specifically, if the SNR for each received signal is  $\gamma = 0$  dB then  $\gamma_{eq} = -1.76$  dB. A 10 dB increase in equivalent SNR is  $\gamma_{eq} = 8.24$  dB, which corresponds to  $\gamma \approx 8.5$  dB. It is conjectured that although this result is only specifically applicable to white data, it should also be suggestive for the signal of interest.



**Figure 1:** Magnitude complex correlation function, OOK and FSK.  $f_s = 312.5\text{ kHz}$ ,  $\text{SNR} = 60\text{ dB}$ ,  $f_b = 5\text{ kHz}$ .



**Figure 2:** Magnitude complex correlation function, OOK and FSK.  $f_s = 312.5\text{ kHz}$ ,  $\text{SNR} = -9\text{ dB}$ ,  $f_b = 5\text{ kHz}$ .

frequency deviation equal to 5, 10, 20 and 40 kHz respectively. For the current simulations with  $f_b = 5$  kHz,  $B = 10$  kHz, evaluation of equation (27) suggests a relative accuracy equal to approximately  $\sigma_{T,OOK}/\sigma_{T,2FSK} = 2, 3.6, 7$  and  $13.9$  respectively. Quantitative consistency between the simulation results and theory is not clearly evident, although the trends are common. The reason for this inconsistency is due to the assumption made in the theoretical development, that the signal has a bandlimited white power spectrum. Just as was identified and resolved in section 2.2 for the study of real, passband TDOA estimation, the degree of improvement in TDOA accuracy is improved substantially when the signal has a non-white power spectrum.

Both the approximate theory and simulation results show a potentially substantial improvement to the rms TDOA estimation accuracy through using an FSK modulation in preference to OOK. For the simulation case of Figure 4, when the SNR exceeds around +5dB, a significant improvement is evident for all frequency deviations considered. However, if it is desirable to ensure a performance improvement across all SNR scenarios examined in the Figure 4 simulations (that is, down to as low as -3.5 dB), an FSK modulation with  $f_0 = f_b = 5$  kHz frequency deviation can be used.

The results from Figure 4 can be used to infer the performance of the original scenario of interest, with an integration period of 1.5 seconds by translating the horizontal axis by 8.5 dB. That is, the performance for the 1.5 second scenario at 0 dB SNR should be equal to that in Figure 4 at +8.5 dB. For greater SNR levels, performance equivalence will be achieved at an SNR difference tending to 10 dB, but at lower SNR, the difference will be less<sup>8</sup>.

## 4 Conclusion

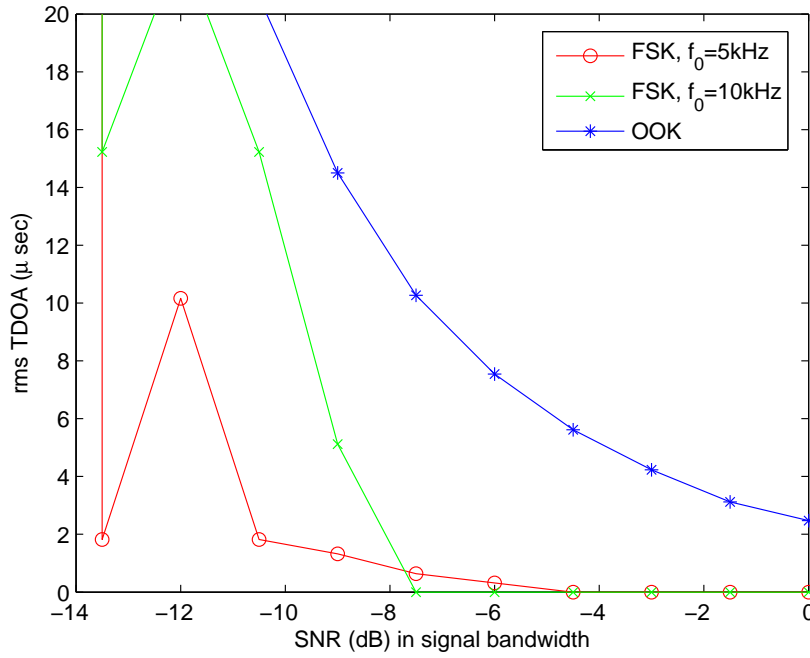
This report has explored the potential for achieving relative TDOA accuracy improvement through using FSK signalling in lieu of OOK. It has been shown through both analysis and simulation that a significant improvement in accuracy can be achieved at an appropriately high SNR. The improvement in the rms TDOA error can exceed an order of magnitude for signals and conditions of practical interest.

## Acknowledgements

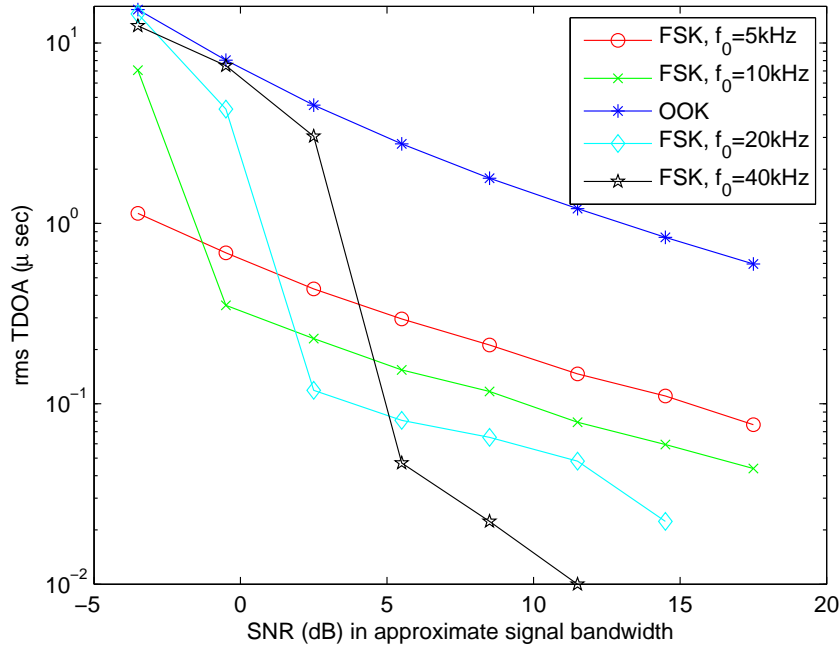
The authors would like to acknowledge each of Don Simkins, Andrew Wiseman, Adrian Caldwell and Michelle Chamalaun for the various informative discussions surrounding this work.

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<sup>8</sup>The accuracy for 1.5 second integration at +10 dB corresponds to the accuracy at approximately +19.5dB with 0.15 second integration. The accuracy for 1.5 second integration at -10 dB corresponds to the accuracy at approximately -4dB with 0.15 second integration.



**Figure 3:** rms TDOA for OOK and FSK modulation.  $f_s = 312.5\text{kHz}$  and  $f_b = 5\text{kHz}$ . Note: SNR is measured within signal bandwidth (for FSK this is  $\pm f_b$  bandwidths around each of  $f_c \pm f_0$ )



**Figure 4:** rms TDOA for OOK and FSK modulation.  $f_s = 10\text{MHz}$ ,  $f_b = 5\text{kHz}$  and  $L = 0.15\text{sec}$ . Note: SNR is measured within approximate signal bandwidth (for FSK, this is  $\pm f_b$  bandwidths around each of  $f_c \pm f_0$ )

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## Appendix A Alternative exploitation of correlation phase

### A.1 Improved Complex Baseband TDOA Estimation

The TDOA estimation benefits of real passband processing can be also captured in the complex baseband domain by exploiting the phase information within the complex correlation function. There are two approaches to this:

**Approach 1** Define

$$\tilde{R}(\tau) = R(\tau) \exp(-j2\pi(f_c - f_I)\tau) \quad (\text{A1})$$

$$= \exp(j2\pi f_c(T - \tau)) \int_t s(t - \tau) s^*(t - T) dt, \quad (\text{A2})$$

where  $R(\tau)$  is defined in equation (11).

Estimation of the TDOA is then via:

$$\begin{aligned} \hat{T} &= \arg \max_{\tau} \Re[\tilde{R}(\tau)] \\ &= \arg \max_{\tau} [\cos(2\pi f_c(T - \tau)) \int_t a(t - \tau) a(t - T) + b(t - \tau) b(t - T) dt \\ &\quad + \sin(2\pi f_c(T - \tau)) \int_t a(t - \tau) b(t - T) - b(t - \tau) a(t - T) dt]. \end{aligned}$$

**Approach 2** This approach involves two steps:

(i) estimate TDOA from the magnitude of the complex-baseband correlation function:

$$\hat{T}_1 = \arg \max_{\tau} |R(\tau)|. \quad (\text{A3})$$

(ii) Define:

$$\phi_1 = \text{phase}[R(\hat{T}_1)] \quad (\text{A4})$$

$$T_e = T - \hat{T}_1. \quad (\text{A5})$$

Then

$$\phi_1 = 2\pi(f_c T - f_I \tau_1) + n2\pi, \text{ where } n = 0, \overset{+}{-} 1, \dots \quad (\text{A6})$$

$$= 2\pi((f_c - f_I)\tau_1 + f_c T_e) + n2\pi. \quad (\text{A7})$$

Hence, estimation of  $T_e$  may be obtained via:

$$\hat{T}_e = (\phi_1/(2\pi) - n) - (f_c - f_I)\hat{T}_1)/f_c \quad (\text{A8})$$

leading to the final TDOA estimate:

$$\hat{T} = \hat{T}_1 + \hat{T}_e. \quad (\text{A9})$$



It should be acknowledged that both of these approaches are susceptible to the  $n/f_c$  estimation ambiguities also inherent to the real passband processing approach. It should also be pointed out that both of these complex baseband approaches above, and the real IF approach in Appendix A.2, require a sampling rate  $f_s \geq 2f_c$  to obtain the phase related TDOA estimation improvements.

## A.2 Real IF TDOA Estimation

The improved TDOA estimation accuracy obtainable from the (real and complex processing) methods in Appendix A.1 can be also obtained with processing the real signals down converted to a suitable intermediate frequency (IF).

$$\begin{aligned} u_1(t) \longrightarrow x_1(t) &= a(t) \cos(2\pi f_I t) - b(t) \sin(2\pi f_I t) \\ u_2(t) \longrightarrow x_2(t) &= a(t - T) \cos(2\pi(f_I t - f_c T)) - b(t - T) \sin(2\pi(f_I t - f_c T)). \end{aligned}$$

Define:

$$\begin{aligned} \bar{x}_1(t - \tau) &= a(t - \tau) \cos(2\pi(f_I t - f_c \tau)) - b(t - \tau) \sin(2\pi(f_I t - f_c \tau)) \\ &= x_1(t - \tau) \cos(2\pi(f_c - f_I)\tau) + x_{1H}(t - \tau) \sin(2\pi(f_c - f_I)\tau) \\ \text{where } x_{1H}(t) &= \text{Hilbert transform}[x_1(t)] \\ \bar{R}(\tau) &= \int_t \bar{x}_1(t - \tau) x_2(t) dt. \end{aligned}$$

Improved TDOA estimation is then obtained via:

$$\begin{aligned} \hat{T} &= \arg \max_{\tau} \bar{R}(\tau) \\ &= \arg \max_{\tau} [\cos(2\pi f_c(T - \tau)) \int_t a(t - \tau) a(t - T) + b(t - \tau) b(t - T) dt \\ &\quad + \sin(2\pi f_c(T - \tau)) \int_t a(t - \tau) b(t - T) - b(t - \tau) a(t - T) dt]. \end{aligned}$$

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